Fibonacci Pitch Sets

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0.0 Introduction

Although the Fibonacci series and its related sequences have long attracted the attention of composers and theorists, their application to pitch structure has not yet been fully understood. In his essay *Duality and Synthesis in the Music of Béla Bartók*, Ernö Lendvai discusses how Bartók may have used the Fibonacci series in the pitch structure of his *Sonata for Two Pianos and Percussion*. Jonathan Kramer, in his article *The Fibonacci Series in Twentieth-Century Music*, briefly discusses its application to pitch structure by several composers such as Bartók, Walker and Schillinger. The most recent article to appear on the subject was by Howard Wilcox, entitled *Generating Fibonacci Sequences of Pitch Classes*. All these writings leave questions about its possible application to pitch structure. This essay will attempt to create a thorough application of Fibonacci sequences to pitch structure, and will be restricted to Fibonacci pitch sets based on a twelve pitch, equally tempered scale, although interesting results can be achieved using other tunings and summation sequences, as a future essay will show.

1.0 Fibonacci Pitch Class Sets

Without a modulus, the sequence would ascend or descend infinitely, quickly out of the range of human hearing. Using mod 12, actual Fibonacci spacing is lost on the large scale, but the Fibonacci property that each succeeding pitch class is the sum of the two previous pitch classes is preserved even as the sequence passes through the mod 12 prism. Reducing the sequence to pitch classes creates a recurring sequence with a period of 24. Note that the same linear ordering of pitch classes would occur in actual Fibonacci spacing:

(13 21 34 55 89 ....)

0 1 1 2 3 5 8 1 9 10 7 5 0 5 5 10 3 1 4 5 9 2 11 1 (0 1 1 ....)

C C# C# D Eb F Ab C# A Bb G F C F F Bb Eb C# E F A D B C#

This sequence can be inverted:

0 11 11 10 9 7 4 11 3 2 5 7 0 7 7 2 9 11 8 7 3 10 1 11 (0 11 11 ....)

C B B Bb A G E B Eb D F G C G G D A B Ab G Eb Bb C# B

The defining relation of these sequences is expressed $f_n + f_{n+1} \equiv f_{n+2}$ (mod 12), read $f_n + f_{n+1}$ is congruent to f_{n+2} modulo 12, where f_n denotes the nth member of the sequence. From here forth these sequences will be discussed as ordered pitch class sets defining the cardinality of the set by the period of the sequence.

1.1 Generating the Collection

For modulo 12, there are a finite number $12^2 = 144$ possible ordered dyads $\{f_n, f_{n+1}\}$ including identities. The sequence is periodic because any ordered dyad determines the entire sequence. These facts enable us to create a fixed number of sequences that contain all 144 ordered dyads. The sets above represent 48 ordered dyads, leaving 96 ordered dyads to complete the possible 144. A next sequence can be derived through the multiplication of the Fibonacci sequence by 2:

0 2 2 4 6 10 4 2 6 8 2 10 0 10 10 8 6 2 8 10 6 4 10 2 (0 2 2 ....)

The inversion of this set occurs in the middle of the sequence {0, 10, 10, …}. Seven more unique sequences can be generated by multiplication or addition operations on the original sequence to complete the 144 possible ordered dyads. While it is useful to list sets in the conventional format shown above, the most accurate representation of the sets is that of pitch class cycles. The cyclic representation proves to be a useful tool; this paper will make use of both representations. Table 1 shows all possible Fibonacci pitch class sets in both cyclic and conventional format. All these sequences can be generated from f1 through multiplication operations, except l1. One can derive l1 from f1 through the following addition operation:

f1		1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11
			+																						
f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1	0
			=																						
l1		2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11

The resulting sequence is the Lucas sequence {2, 1, 3, 4, 7, 11, 18, 29, 47, …}.[1] All possible Fibonacci sequences modulo 12 occur in these sets. The Evangelists' sequence,[2] another well known Fibonacci sequence beginning {3, 2, 5, 7, 12, 19, 31, 50, …}, can be seen in the inverted form of f1.

1.2 Some Basic Properties of the Collection

While individual Fibonacci pitch class sets bear minimal resemblance to twelve-tone sets, the collection of Fibonacci sets bears some resemblance to the all-interval set in twelve-tone music. Here not only does every interval and pitch class occur an equal amount of times; every possible adjacent combination of intervals occurs as well. Because the intervals between successive Fibonacci numbers themselves form an adjacent Fibonacci series, these sets can be viewed simultaneously as pitch class sets and interval class sets. In this paper the term residue will be used to refer to pitch and interval class simultaneously.[3]

In *The Structure of Atonal Music*, Allen Forte gives the criterion that a set of pitch classes must have no repeated pitches.[4] But repeated pitch classes have great significance in Fibonacci pitch sets; this system assumes the presence of twelve intervals. The fact that repeated pitch classes are part of the set suggests that the composer should totally avoid other pitch class repetition - as one might do in a twelve-tone context - unless indicated in the set. Any deviation from the ordered sets would distort the Fibonacci relationship between linear adjacencies.

1.3 A Caution

Before proceeding to the analysis portion let me clarify the fact that transposing these sets by n where $n \neq 0$ distorts our notation of the Fibonacci property; for example look at the set A = T(f1, 1):

A 1 2 2 3 4 6 9 2 10 11 8 6 1 6 6 11 4 2 5 6 10 3 0 2 1

This sequence does not appear to contain Fibonacci properties because it is not in the normal form shown in table 1. Only analysis of the intervals between the successive pitch classes would reveal a Fibonacci sequence. While fixed pitch numeration has many advantages, it can distort the notation of the Fibonacci property.

2.0 Analysis

We will begin the analysis portion of this essay with a more general consideration of the properties of Fibonacci sequences under moduli while using examples from the modulo 12 pitch sets. A definition is helpful at this point.

DEFINITION. *Let n = some integer. Let f_n denote a Fibonacci sequence of integers (not under a modulus) where the defining relation is $f_n + f_{n+1} = f_{n+2}$ for the successive terms of the sequence: $f_0, f_1, f_2, \ldots, f_n, \ldots$, and let $f_0 = x$ and $f_1 = z$. Let m = some integer used as a modulus.*

2.1 Periods, Restricted Periods, and Multipliers

The period of a Fibonacci sequence under a modulus is expressed by the smallest integer n that satisfies $f_n \equiv x \pmod{m}$ and $f_{n+1} \equiv z \pmod{m}$.

f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
f5	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11

The period of f1 and f5 is 24. The reader will notice that the period of the other sets differs from those discussed above; we will investigate the periods of the other sets shortly. First let us establish some more basic properties. Notice how the 24 residues in sets f1 and l1 form two subsets of 12; the second subset is equivalent to the first multiplied by 5. This set is said to have a restricted period of 12. A restricted period is represented by the smallest integer n that satisfies $f_n \equiv s(x)$ and $f_{n+1} \equiv s(z) \pmod{m}$ for some integral s; s is called the multiplier and in the case of f1, l1, and f2, $s = 5$ and $n = 12$. A Fibonacci sequence under a modulus always contains 1, 2, or 4 of these restricted subsets.[5] f0, f3 and f6 have 1 restricted subset (they have no restricted period), while the rest of the sets contain 2 restricting subsets. When a set has no restricted period, it is equivalent to itself when multiplied by 5. This shows the special significance of the number 5 for all modulo 12 sets; 5 is the smallest number (besides 1) relatively prime to 12, and it can be seen that the multiplier for all the Fibonacci modulo 12 sets is $s = 5$.

2.2 Invertibility

Any Fibonacci sequence f_n modulo m can be inverted to form a sequence m – (f_n modulo m) that also contains the Fibonacci property.[6] The inverted counterpart of a set f*n* will be referred to as f*n*' when distinction is necessary. f1 and l1 have a period of 24, but can be inverted to form a unique set; thus f1 and l1 contain 48 residues each. f2 also has a period of 24, but this set maps onto its second restricted subset when inverted; it consequently contains 24 residues. f3 has a period of 6, and inverts to form a unique set; this set contains 12 residues. f4 maps onto its second restricted subset when inverted; it contains 8 residues. f6 contains 3 residues, and maps onto itself when inverted. f0 maps onto itself when inverted and contains only 1 residue. Table 2 shows the total occurrences of each residue class in the Fibonacci sets.

2.3 Prime Sets and Subsets

The absence of certain residue classes in f1 and f5 is a direct result of the modulus used; modulo 12 is said to be a defective modulus,[7] as the Fibonacci sequence doesn't contain a complete system of residues modulo 12.[8] Residue class 6 is absent in f1. This results from the fact, and could be regarded as proof of the fact, that every Fibonacci number divisible by six is also divisible by twelve. Residues 0 and 9 are absent in l1. This proves that no Lucas number is divisible by twelve, and that no Lucas number is divisible by twelve with a remainder of nine. Because residue 0 is not a member of l1, it is the only set with no repeated pitch or interval classes.

In f1 and l1 the starting values and the modulus are relatively prime; $(f_0, f_1, m) = 1$, or more comprehensively $(f_n, f_{n+1}, m) = 1$. We will refer to f1 and l1 as the prime sets. These prime sets produce the same subsets when multiplied by the same number. Earlier, we derived f2 as (f1* 2). But f2 can also be derived as (l1 * 2). The absence of certain pitch classes in sets f2, f3, f4, and f6 stems from a different cause from that in f1 and l1; these omissions result from the fact that their starting integers are not relatively prime to the modulus; $(f_0, f_1, m) = k$ where an integer $k \geq 2$. Any ordered dyad in the sequence $\{f_n, f_{n+1}\}$ and the modulus m can be reduced. We will refer to such sets as subsets of the prime sets. This explains why the periods, as well as other properties of these subsets, differ from those of f1 and l1. For example, when analyzing f2, where $f_0 = 0$ and $f_1 = 2$ under a modulus $m = 12$, we are actually analyzing the properties of a congruent sequence where $f_0 = 0$ and $f_1 = 1$ and the modulus $m = 6$. It is worth viewing these sequences in their prime form; all subsets in the modulo 12 collection are shown in their prime form in table 3.

The next equally tempered subset of the 12 tone scale is the 6 tone scale. If we used only 6 pitches per octave f2 might have been called f1 (mod 6), a prime set in the modulo 6 universe. Under modulo 6, the Fibonacci and Lucas sequences merge into the same sequence; the Lucas sequence begins half way into the restricted period (after 6 terms) of the Fibonacci sequence. All possible residues appear; so the Fibonacci sequence contains a complete system of residues modulo 6. With 6 equally tempered pitches per octave, it would also be possible to create subsets such as f4, as well as f0 and f6. These sets together complete the $6^2 = 36$ possible ordered dyads under modulo 6. The diagram in table 3 shows relations between subsets and supersets in the mod 12 collection. The mod 6 collection can be seen including, and extending from f2, which is itself a subset of f1. f4 should be thought of not only as f1 (mod 3), but also as f2 (mod 6), a subset in the mod 6 collection. f3 is not a subset of f2 because it would not be possible to create f3 with 6 equally tempered pitches per octave.

Another equally tempered subset of the 12 tone scale is the 4 tone scale; it contains $4^2 = 16$ possible ordered dyads. Table 3 shows how the Lucas and Fibonacci sequences remain unique under modulo 4, while they merge into the same sequence under moduli 6, 3, 2, and 1. f6 is congruent to a Fibonacci sequence where $f_0 = 0$ and $f_1 = 1$ and the modulus $m = 2$. f6 and f0 are the only sets having a period that is an odd number. All Fibonacci sequences modulo m where $m > 2$ have a period that is an even number.[9] All the proper divisors of 12, moduli 6, 4, 3, 2, and 1, prove not to be defective. While every multiple of a defective modulus is defective, it does not follow that some proper divisor of that modulus would be defective.[10]

2.4 Semi-Symmetry

Table 4 shows semi-symmetry in pitch class cycles f1 and l1. Each meridian connects identical pitch classes on the opposite side of the set. The semi-symmetrical nature of Fibonacci sequences is most easily understood when the sequences are viewed extended into the negative numbers f_{-n}. Notice the alternation of negative and positive numbers in the Fibonacci sequence:

...	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	34 ...

The additive identity 0 acts as an axis point, causing every other residue to be equivalent to its counterpart on the opposite side of the set. The residues in between these are the inversions of their counterparts. If f_{2n} represents some even positioned member in a Fibonacci sequence f_n (mod m) and $f_0 = 0$, then $f_{2n} \equiv m - f_{-2n}$ (mod m) and $f_{2n-1} \equiv f_{-2n+1}$ (mod m). The Lucas sequence has a similar property:

...	-76	47	-29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	47	76 ...

In the Lucas sequence, 2 acts as the axis instead of 0, and the property is reversed so that if $l_0 = 2$ and $l_1 = 1$, then $l_{2n} \equiv l_{-2n}$ (mod m) and $l_{2n-1} \equiv \mathrm{m} - l_{-2n+1}$ (mod m).

2.5 Interval Cycles

The interval cycles are another member of the pitch class cycle family; their arithmetic and equal tempered nature make them a useful tool in the analysis of Fibonacci pitch sets. The interval cycles are shown in table 5.[11] The interval cycle(s) that a pitch class set belongs to can be determined by any ordered dyad occurring in that pitch class set. Elements of f1 and l1 belong to cycles C1 and C5; we have noted individual omissions. Elements of f2 are found in $C2_0$. Elements of f3 are found in $C3_0$. Elements of f4 are found in $C4_0$. Elements of f6 are found in $C6_0$. This separation of sonorities is of great audible significance. Also of significance is that every third pitch of f1 and l1 belongs to $C2_0$, the pitches in between belonging to $C2_1$:[12]

f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
l1	6	1	7	8	3	11	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7

The alert reader may recognize this pattern of elements belonging to …, $C2_0$, $C2_1$, $C2_1$, $C2_0$, $C2_1$, $C2_1$, …, or more generally {…, *x*, *z*, *z*, *x*, *z*, *z*, …}, as a familiar Fibonacci pattern. It can be seen that a reduction of these sets can be made by generalized Fibonacci sequences of interval cycles; $C2_0 + C2_1 = C2_1$ and $C2_1 + C2_1 = C2_0$. The subscripts form a Fibonacci sequence modulo 2 {…, 0, 1, 1, 0, 1, 1, …}. If a set begins $C2_0 + C2_0 = C2_0$, then the whole set is made up of elements of $C2_0$ such as in f0, f2, f4, and f6. Notice also that every fourth member of f1 and l1 is a member of $C3_0$ (i.e. is divisible by 3); all the pcs in between belong to the octatonic set $C3_{1,2}$. We can establish a rule that if $f_0 \in Cn_x$ and $f_1 \in Cn_z$, we use a Fibonacci sequence modulo n and start with the values of $x + z$ to form a generalized sequence. Table 6 shows all possible generalized Fibonacci interval cycle sequences.

2.6 Invariances

Each ordered dyad, excluding identity dyads, is contained in an invariant trichord. The term invariant is used in this section in regard to content; ordering will always be unique. We know from the defining relation and the commutative property of addition that whenever an ordered dyad occurs, the succeeding term will be the same as it would had the dyad been in the reverse order. It is the ordering that is of importance in determining the term that will succeed this trichord. The identity dyads are the same when ordering is reversed. Trichords beginning with identity dyads $\{x, x, y\}$ have no invariant counterpart, with one exception; the Fibonacci trichord {6, 6, 0} intersects with its two other invariant counterparts, and has 3 possible Fibonacci orderings. This makes for a total of 11 trichords containing no invariant counterpart. There are 11 more pairs of intersecting invariant trichords; they take the form $\{x, 0, x\}$ and $\{0, x, x\}$ with $1 \leq x \leq 11$. There are 55 remaining pairs of invariant trichords that do not intersect; they take the form $\{x, y, z\}$ and $\{y, x, z\}$. Table 7 shows the relations between all invariant trichords in the prime sets. The beginning of each ordered dyad in this table is marked by a dot above the first term of the dyad. A line, or an arch if it is in the same sequence, is drawn from the dot above one dyad to the dot above its invariant counterpart.

Invariant tetrachords occur in these sets as well. Table 8 shows all invariant Fibonacci tetrachords in the prime sets. Any Fibonacci tetrachord whose second element is 4 or 8 has 2 invariant counterparts. It can be seen that each case of a tetrachord occurring in three different places involves a subset $C4_x$ and that the second element is always either 4 or 8. The second element of any Fibonacci tetrachord is involved in all addition that takes place to complete that tetrachord. The contents of $C4_x$ are, by the nature of the set, separated by intervals of 4; adding the second element 4 (or 8) to any first element from $C4_x$ will result in another different pitch class in $C4_x$. Adding 4 (or 8) again to the resulting number completes the third element in the set $C4_x$. The 4 or 8 behaves as an additive identity for $C4_x$. Any tetrachord beginning 6 has 1 invariant counterpart. For example in l1 the tetrachords {6, 1, 7, 8} and {6, 7, 1, 8} occur, in f2 the tetrachords {6, 2, 8, 10} and {6, 8, 2, 10} occur. There are no cases of invariance above the tetrachordal level.

There is another phenomenon that deserves mention here. Since $f_{n+2} - f_{n+1} = f_n$, it follows that the term preceding any ordered dyad (or invariant trichord) would be the inverse of the term that would have occurred had the dyad been in reverse order. For example we have {1, 1, 2, 3} and {11, 2, 1, 3}, or {9, 4, 1, 5} and {3, 1, 4, 5}. While these are not invariant tetrachords in the traditional sense, they should be noticed and thought of as a broader type of invariance.

3.0 Inverse Fibonacci Pitch Class Sets

New linear possibilities arise out of inverse Fibonacci pitch class sets. Inverse Fibonacci is a sequence where for $\ldots, f_n, f_{n+1}, f_{n+2}, \ldots$, the defining relation is $f_n - f_{n+1} = f_{n+2}$. Each successive term is the difference of the previous two. The inverse Fibonacci sequence begins {0, 1, -1, 2, -3, 5, -8, 13, -21, …}. Inverse Fibonacci pitch class sets, represented by –f*n*, are equivalent to the retrogrades of the Fibonacci sequences mod 12:

-f1	0	1	11	2	9	5	4	1	3	10	5	5	0	5	7	10	9	1	8	5	3	2	1	1

Notice what results from the semi-symmetrical property when the sequences are used in retrograde; all the odd positioned members of the sequence f_{2n-1} remain unchanged, and the even positioned members f_{2n} are inverted from the original sequence f1. Table 9 shows all inverse Fibonacci pitch class sets. Inverse pitch class cycles are identical to the original cycles but read counterclockwise.

4.0 Fibonacci Harmony

After having considered the Fibonacci sequences from a primarily linear (melodic) standpoint, I now shift focus to the harmonic properties of the sequence. For any Fibonacci sequences f_x and l_y, there exists a sequence $h_z = f_x - l_y$; sequence h_z contains the Fibonacci property.[13] Here is an example from the mod 12 collection:

f1	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1	0
	-																							
f2	0	2	2	4	6	10	4	2	6	8	2	10	0	10	10	8	6	2	8	10	6	4	10	2
	=																							
hf1'	1	11	0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10

Subtracting in one-to-one correspondence, the example above can be expressed {f1: 1, 1, …} – {f2: 0, 2, …} = { hf1': 1, 11, …}. hf*n* is a Fibonacci interval class set expressed harmonically. Because there are 144 ordered dyads, there are 144 possible unique harmonizations of any given set. Consider the first dyad in f1 shown above {f1: 1, 1, …}, this dyad can be paired with any of the 144 ordered dyads {f*n*: *x*, *y*, …} to create a unique set {hf*z*: *v*, *w*, …}. If $x = 1$ and $y = 1$ (the same set mapped to itself), then the resulting set would be {f1: 1, 1, …} – {f1: 1, 1, …} = {hf0: 0, 0, …}. If $x = 0$ and $y = 0$, this identity set f0 combines to always give us our original sequence {f1: 1, 1, …} – {f0: 0, 0, …} = {hf1: 1, 1, …}. The fact that there are 144 possible Fibonacci harmonizations of a given set implies that any given set hf*n* can be expressed 144 different ways. Let us take an example. The example first given rendered the set {hf1' : 1, 11, …}. Among the 143 other possibilities, this harmonic sequence could also have been created by {f2: 2, 2, …} – {l1: 1, 3, …} = {hf1' : 1, 11, …}. It is also clear that no matter how many Fibonacci sets are used in one-to-one correspondence, the relationship between each set is Fibonacci. These properties apply to inverse Fibonacci sets as well, but no type of Fibonacci harmony will result in the pairing of a set f*n* with an inverse set –f*k*.

5.0 Nonmodular Fibonacci Pitch Sets

Nonmodular Fibonacci pitch sets have already been explored to some degree by numerous composers and theorists.[14] They are of special interest because of their unique sonority, and golden section proportions. These sets can be indexed by the first two pitch integers *n* and *m* that are used, as they define the pitches that succeed. It is convenient to use the symbol $F_{n,m}t$ to represent nonmodular Fibonacci pitch sets where *t* denotes the cardinality of the set. Inverted (descending) Fibonacci sets are denoted by $F_{-n,-m}t$; so the set $F_{-1,-2}8$ = {-1, -2, -3, -5, -8, -13, -21, -34}. In normal form *n* and *m* are less than 12. It is also possible to extend these pitch sets into negative numbers f_{-n}, such as the set $F_{-21,13}15$ = {-21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8}. When sets are extended into negative Fibonacci numbers, the alternation of negative and positive in the ordering results in a Fibonacci "wedge" that closes in. Table 10 shows the normal forms for set f1. The table is organized by intersections of interval and pitch classes in the sets, which can be deduced easily from the pitch class sets initially constructed. These pitch sets can only be so large, as they quickly approach extreme limits of our hearing as the cardinality increases. $F_{0,0}t$ is the exception to this rule and can have infinite cardinality. A range of 112 semitones was fixed to the sets on both sides of the sequence f_n and f_{-n}; this number seems appropriate, as this spans a frequency range of approximately 30 – 16,000hz.

5.1 Inverse Fibonacci Pitch Sets

It is also possible to construct pitch sets using inverse Fibonacci sequences. It is helpful at this point to view an inverse Fibonacci pitch class set using a variation of mod 12 where each residue must have an absolute value less than 12:

-f1 0 1 -1 2 -3 5 -8 1 -9 10 -7 5 0 5 -5 10 -3 1 -4 5 -9 2 -11 1

This sequence is, of course, identical in pitch class content with the original inverse sequence. It is convenient to use this form of the sequences to generate the normal forms for inverse Fibonacci pitch sets. Inverse pitch sets are indexed by the first two pitch integers *n* and *m* that occur. Inverse Fibonacci pitch sets are expressed in the form $-F_{n,m}t$; *n* and *m* can be a negative or positive pitch integer, and in normal form must have an absolute value less than 12. So the set $-F_{8,-7}6$ = {8, -7, 15, -22, 37, -59}. These sets can be extended into negative numbers, such as $-F_{5,3}10$ = {5, 3, 2, 1, 1, 0, 1, -1, 2, -3}. Table 11 shows inverse Fibonacci pitch sets generated from –I1 in normal form within the same limit of 112 semitones. These sets can be inverted by flipping every integer to its additive inverse, so $-F_{-5,-3}10$ = {-5, -3, -2, -1, -1, 0, -1, 1, -2, 3}. The inverse Fibonacci "wedge" extends outward. These sets cause convergence on the negative golden section (1 – .61803…) on both sides of the axis 0. The best example of this is the set $-F_{0,1}11$ {0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55}; the last two negative terms in the set -21/-55 = 1 – .618, and the last two positive terms 13/34 = 1 – .617647….

6.0 Beyond Fibonacci Modulo 12

In this essay I have attempted, using twelve tone equal temperament, to create a thorough application of Fibonacci sequences to pitch structure. This essay, however, leaves much to be explored in the application of summation sequences to pitch structure. In my next essay I will explore Fibonacci sequences using temperaments other than twelve tone, as well as exploring a summation sequence of greater complexity: the Tribonacci sequence, which begins {0, 0, 1, 1, 2, 4, 7, 13, 24, 44, …}.

Notes

I would like to thank Alan Fletcher for his many valuable suggestions during the preparation of this essay.

1 This sequence was named after Edouard Lucas (1842 – 1891), a French mathematician who gave the Fibonacci sequence its name and also suggested using Fibonacci's name for what is now known as the Lucas sequence {2, 1, 3, 4, 7, 11, 18, 29, 47, …}. This sequence has many fascinating relations with the Fibonacci sequence, and also has many unique properties not found in the Fibonacci sequence. One property worth mentioning here is the fact that $f_{n-1} + f_{n+1} = l_n$ for all n where f_n = nth Fibonacci number, and l_n = the nth Lucas number. This relation was used to derive l1 from f1. For an excellent introduction to the properties of both Fibonacci and Lucas numbers, the reader could do no better than to refer to Hoggatt 1969.

2 The Evangelists' sequence was named so because of the occurrence of the numbers {2, 5, 7, 12} in the Evangelists' account of Jesus feeding the multitude (Matthew 14: 17-20, John 6: 9-13, Matthew 15: 34-37). One of its many interesting relations to the Fibonacci sequence is $e_n = f_{n+3} - f_{n-2}$ for all n where e_n = the nth number in the Evangelists' sequence and f_n = the nth Fibonacci number.

3 Those readers familiar with modular arithmetic will recognize the concept of a residue class as the congruence class of all integers with the same remainder upon divisibility by the modulus m. Congruence is an equivalence relation. If $a \equiv b$ (modulo m), then a and b belong to the same residue class of m. For example, -10, 2 and 14 all belong to the same residue class modulo 12.

4 Forte 1973: 3.

5 See Mandelson 1968: 276. Mandelson refers to these subsets as groups; these subsets are not always groups in the strict mathematical sense. Here the term restricted subset is used to avoid any confusion.

6 PROOF. Consider a Fibonacci sequence F_n (mod m), beginning $f_0, f_1, f_0 + f_1$ (mod m), …, there exists a sequence $m - [F_n \text{ (mod } m)]$; $m - f_0$, $m - f_1$, $m - (f_0 + f_1)$ (mod m), …. And it can be seen that the first two terms $(m - f_0) + (m - f_1) = 2m - (f_0 + f_1) \equiv m - (f_0 + f_1)$ (mod m). So the sequence $m - [F_n \text{ (mod } m)]$ is a Fibonacci sequence.

7 For a further discussion of defective moduli, and moduli for which the Fibonacci sequence contains a complete system of residues see Shaw 1968, Bruckner 1970, and Burr 1971.

8 A complete system of residues modulo *m* is a set that contains every possible residue class of *m*. An example of a complete residue system modulo 12 is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. Another example is {-5, 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50}.

9 Wall 1960 Theorem 4.

10 Shah 1968 Theorem 2.

11 George Perle introduced the abbreviation Cn_x to represent the interval cycles where *n* designates the interval of the cycle and *x* designates the smallest pitch integer of the cycle, see Perle 1985: 198-200.

12 This property can be seen in the Fibonacci sequence in the fact that every third Fibonacci number is a multiple of 2, every fourth Fibonacci number a multiple of 3, every fifth a multiple of 5, every sixth a multiple of 8, every seventh a multiple of 13, and so on. The Lucas sequence has similar properties; every third Lucas number is a multiple of 2 and that every fourth is a multiple of 3, but the Lucas sequence is more complex than the Fibonacci sequence in this regard; the reader is encouraged to investigate this or see Hoggatt 1969.

13 PROOF. Consider two Fibonacci sequences F_x and L_y with the defining relations $f_x + f_{x+1} = f_{x+2}$, and $l_y + l_{y+1} = l_{y+2}$. We will now define a third sequence U_z by $u_z = f_x - l_y$, $u_{z+1} = f_{x+1} - l_{y+1}$, and $u_{z+2} = f_{x+2} - l_{y+2}$. Now we must prove that $u_z + u_{z+1} = u_{z+2}$, and it is clear that $(f_x - l_y) + (f_{x+1} - l_{y+1}) = (f_x + f_{x+1}) - (l_y + l_{y+1}) = (f_{x+2} - l_{y+2})$. So sequence U_z contains the Fibonacci property.

14 Kramer 1973: 134-141.

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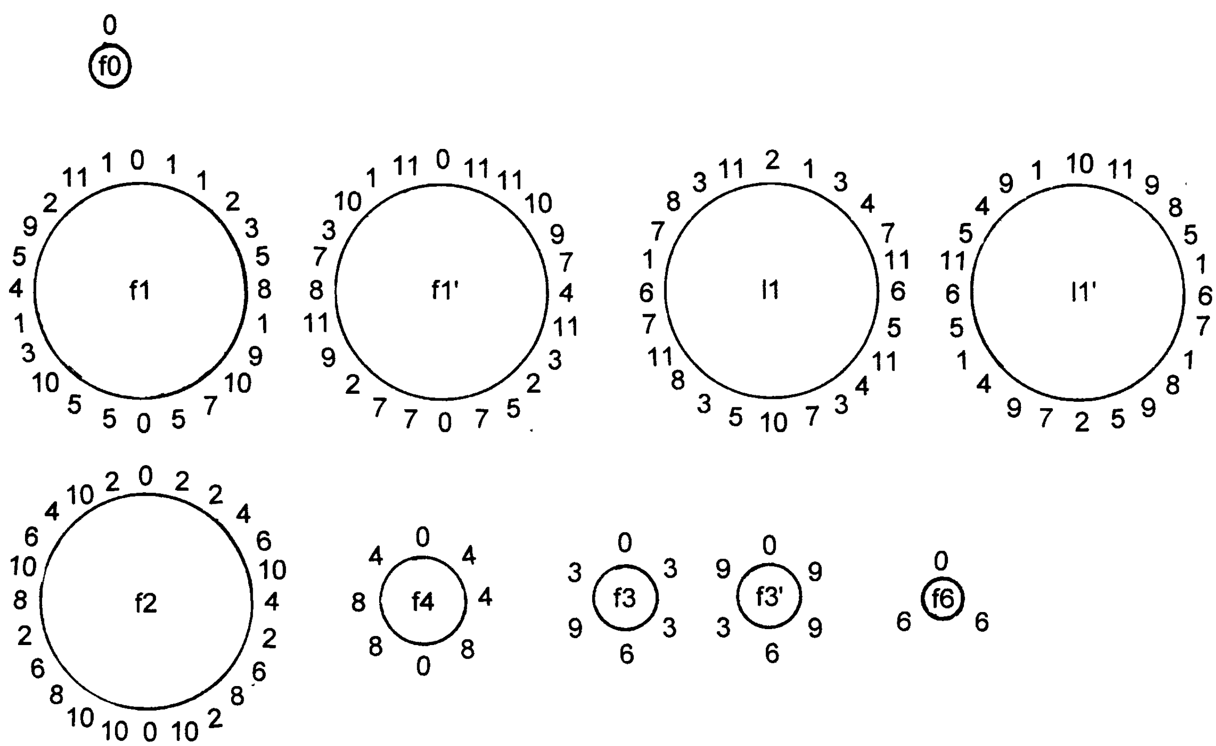
Table 1
Fibonacci Pitch Class Sets

f0	0																							
f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
	0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11
I1	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
	10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1
f2	0	2	2	4	6	10	4	2	6	8	2	10	0	10	10	8	6	2	8	10	6	4	10	2
f3	0	3	3	6	9	3																		
	0	9	9	6	3	9																		
f4	0	4	4	8	0	8	8	4																
f6	0	6	6																					



to accompany pg. 2

Table 2
Total Occurrences of Each Residue *r* in f*n*

r - # of occurrences

r	f0	f1	f1'	I1	I1'
0	1	2	2	0	0
1	0	5	1	2	4
2	0	2	2	1	1
3	0	2	2	4	0
4	0	1	1	2	2
5	0	5	1	2	4
6	0	0	0	2	2
7	0	1	5	4	2
8	0	1	1	2	2
9	0	2	2	0	4
10	0	2	2	1	1
11	0	1	5	4	2

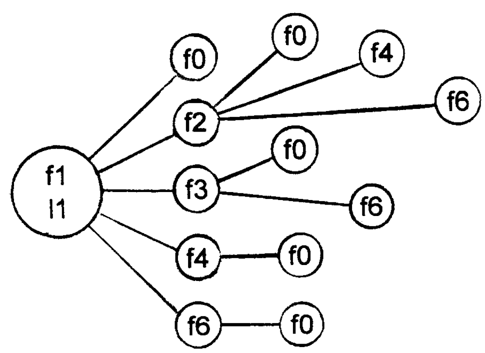
r	f2	f3	f3'	f4	f6
0	2	1	1	2	1
1	0	0	0	0	0
2	6	0	0	0	0
3	0	3	1	0	0
4	3	0	0	3	0
5	0	0	0	0	0
6	4	1	1	0	2
7	0	0	0	0	0
8	3	0	0	3	0
9	0	1	3	0	0
10	6	0	0	0	0
11	0	0	0	0	0

to accompany pg. 5

Table 3
Prime Forms of Mod 12 Subsets

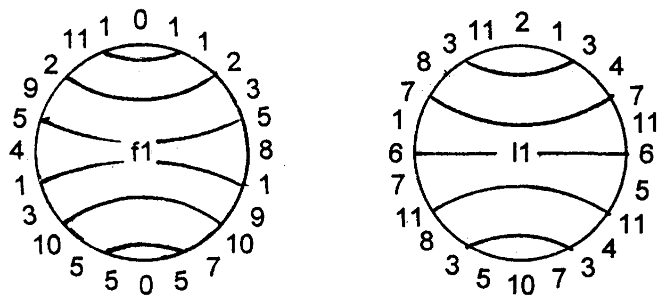
f1 (mod 6)	0	1	1	2	3	5	2	1	3	4	1	5	0	5	5	4	3	1	4	5	3	2	5	1
f1 (mod 4)	0	1	1	2	3	1																		
	0	3	3	2	1	3																		
f1 (mod 3)	0	1	1	2	0	2	2	1																
f1 (mod 2)	0	1	1																					

Mod 12 Subsets and Supersets



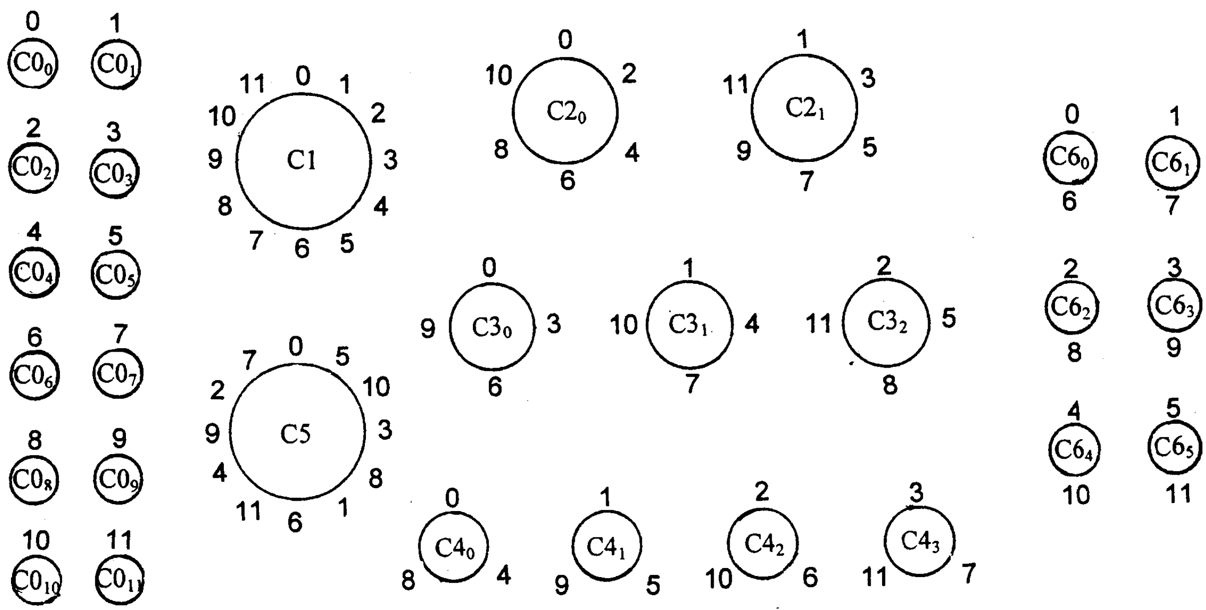
to accompany pg. 6

Table 4
Semi-Symmetry in Pitch Class Cycles



to accompany pg. 6

Table 5
The Interval Cycles



to accompany pg. 7

Table 6
Generalized Fibonacci Interval Cycle Sequences

$C2_0$
$C2_0$ $C2_1$ $C2_1$

$C3_0$
$C3_0$ $C3_1$ $C3_1$ $C3_2$ $C3_0$ $C3_2$ $C3_2$ $C3_1$

$C4_0$
$C4_0$ $C4_1$ $C4_1$ $C4_2$ $C4_3$ $C4_1$
$C4_0$ $C4_3$ $C4_3$ $C4_2$ $C4_1$ $C4_3$
$C4_0$ $C4_2$ $C4_2$

to accompany pg. 8

Table 7
Invariant Trichords in the Prime Sets

f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
	0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11
I1	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
	10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1

I1	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
	10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1
f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
	0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11

to accompany pg. 8

Table 8
Invariant Tetrachords in the Prime Sets

f1	0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
	0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11
I1	2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
	10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1

to accompany pg. 8

Table 9
Inverse Fibonacci Pitch Class Sets

-f0	0																							
-f1	0	1	11	2	9	5	4	1	3	10	5	5	0	5	7	10	9	1	8	5	3	2	1	1
	0	11	1	10	3	7	8	11	9	2	7	7	0	7	5	2	3	11	4	7	9	10	11	11
-I1	2	11	3	8	7	1	6	7	11	8	3	5	10	7	3	4	11	5	6	11	7	4	3	1
	10	1	9	4	5	11	6	5	1	4	9	7	2	5	9	8	1	7	6	1	5	8	9	11
-f2	0	2	10	4	6	10	8	2	6	8	10	10	0	10	2	8	6	2	4	10	6	4	2	2
-f3	0	3	9	6	3	3																		
	0	9	3	6	9	9																		
-f4	0	4	8	8	0	8	4	4																
-f6	0	6	6																					

to accompany pg. 9

Table 10
Fibonacci Pitch Sets

					-55	34	-21	13	-8	5	-3	2	-1	1	**0**	1	1	2	3	5	8	13	21	34	55	89	
				-55	34	-21	13	-8	5	-3	2	-1	1	0	**1**	1	2	3	5	8	13	21	34	55	89		
			-55	34	-21	13	-8	5	-3	2	-1	1	0	1	**1**	2	3	5	8	13	21	34	55	89			
		-55	34	-21	13	-8	5	-3	2	-1	1	0	1	1	**2**	3	5	8	13	21	34	55	89				
	-55	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	**3**	5	8	13	21	34	55	89					
-55	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	**5**	8	13	21	34	55	89						
										-59	37	-22	15	-7	**8**	1	9	10	19	29	48	77					
									-59	37	-22	15	-7	8	**1**	9	10	19	29	48	77						
								-59	37	-22	15	-7	8	1	**9**	10	19	29	48	77							
										-45	29	-16	13	-3	**10**	7	17	24	41	65	106						
									51	-31	20	-11	9	-2	**7**	5	12	17	29	46	75						
									65	-40	25	-15	10	-5	**5**	0	5	5	10	15	25	40	65	105			
								65	-40	25	-15	10	-5	5	**0**	5	5	10	15	25	40	65	105				
							65	-40	25	-15	10	-5	5	0	**5**	5	10	15	25	40	65	105					
						65	-40	25	-15	10	-5	5	0	5	**5**	10	15	25	40	65	105						
										-65	41	-24	17	-7	**10**	3	13	16	29	45	74						
								-50	31	-19	12	-7	5	-2	**3**	1	4	5	9	14	23	37	60	97			
							-50	31	-19	12	-7	5	-2	3	**1**	4	5	9	14	23	37	60	97				
						-50	31	-19	12	-7	5	-2	3	1	**4**	5	9	14	23	37	60	97					
					-50	31	-19	12	-7	5	-2	3	1	4	**5**	9	14	23	37	60	97						
										-62	39	-23	16	-7	**9**	2	11	13	24	37	61	98					
									-62	39	-23	16	-7	9	**2**	11	13	24	37	61	98						
											52	-31	21	-10	**11**	1	12	13	25	38	63	101					
						-55	34	-21	13	-8	5	-3	2	-1	**1**	0	1	1	2	3	5	8	13	21	34	55	89

to accompany pg. 10

Table 11
Inverse Fibonacci Pitch Sets

				76	47	29	18	11	7	4	3	1	**2**	-1	3	-4	7	-11	18	-29	47	-76
			76	47	29	18	11	7	4	3	1	2	**-1**	3	-4	7	-11	18	-29	47	-76	
		76	47	29	18	11	7	4	3	1	2	-1	**3**	-4	7	-11	18	-29	47	-76		
	76	47	29	18	11	7	4	3	1	2	-1	3	**-4**	7	-11	18	-29	47	-76			
76	47	29	18	11	7	4	3	1	2	-1	3	-4	**7**	-11	18	-29	47	-76				
								-58	-37	-21	-16	-5	**-11**	6	-17	23	-40	63				
					99	61	38	23	15	8	7	1	**6**	-5	11	-16	27	-43				
				99	61	38	23	15	8	7	1	6	**-5**	11	-16	27	-43					
							111	68	43	25	18	7	**11**	-4	15	-19	34	-53	87			
					-73	-45	-28	-17	-11	-6	-5	-1	**-4**	3	-7	10	-17	27	-44			
				-73	-45	-28	-17	-11	-6	-5	-1	-4	**3**	-7	10	-17	27	-44				
			-73	-45	-28	-17	-11	-6	-5	-1	-4	3	**-7**	10	-17	27	-44					
							90	55	35	20	15	5	**10**	-5	15	-20	35	-55				
					-107	-66	-41	-25	-16	-9	-7	-2	**-5**	3	-8	11	-19	30	-49			
				-107	-66	-41	-25	-16	-9	-7	-2	-5	**3**	-8	11	-19	30	-49				
							-96	-59	-37	-22	-15	-7	**-8**	11	-9	10	-19	29	-48			
							87	53	34	19	15	4	**11**	-7	18	-25	43	-68				
						-69	-43	-26	-17	-9	-8	-1	**-7**	6	-13	19	-29	48				
							70	43	27	16	11	5	**6**	-1	7	-8	15	-23	38	-61	99	
						70	43	27	16	11	5	6	**-1**	7	-8	15	-23	38	-61	99		
					70	43	27	16	11	5	6	-1	**7**	-8	15	-23	38	-61	99			
							-80	-49	-31	-18	-13	-5	**-8**	3	-11	14	-25	39	-64			
						-80	-49	-31	-18	-13	-5	-8	**3**	-11	14	-25	39	-64				
								-78	-49	-29	-20	-9	**-11**	2	-13	15	-28	43				